Topology

Semestral Examination

20th September 2012

Instructions: All questions carry equal marks. All sets and collections in the questions are assumed to be non-empty!

1. A *subbasis* $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union is X. The topology generated by a subbasis is the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

(a). Prove that the collection of all finite intersections of elements of a subbasis $\mathcal{S}$ forms a basis of the topology generated by $\mathcal{S}$. (Such a basis is said to be generated by a subbasis.)

(b). Give an example of a basis of the real line $\mathbb{R}$ that is generated by a subbasis and an example of one that is not generated by any subbasis.

2.

(a). Define a retraction of a space X onto a subspace A. Prove that a retraction is a quotient map.

(b). Prove that the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = y^3 + xy^2 + x + y$$

is a quotient map.

3. Let X and Y be connected spaces and A and B be proper subsets of X and Y respectively. Prove that $(X \times Y) \setminus (A \times B)$ is connected.

4. Recall that a point x of a space X is called an *isolated point* if $\{x\}$ is open in X.

(a). Prove that a compact hausdorff space without an isolated point must be uncountable.

(b). Does (a) imply that the set of rationals in the compact interval $[0, 1]$ are uncountable? Justify your answer.

5. Prove that a subspace A of the real line $\mathbb{R}$ is compact if and only if every continuous real valued function on A is bounded.